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On CC-comparability invariance of the fixed point property[1]

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Abstract

It is proved that the fixed point property is a comparability invariant for the class of chain-complete ordered sets. As a lemma we derive a canonical decomposition for chain-complete ordered sets.

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1. Introduction

We say two points p_1, p_2 in an ordered set P are *related* or *comparable* ($p_1 \sim p_2$) iff $p_1 \leqslant p_2$ or $p_1 \geqslant p_2$. The *comparability graph* of an ordered set P is the graph $(P, \{\{p_1, p_2\}: p_1 \neq p_2,\ p_1 \sim p_2\})$ that has as its vertices the points of P and contains an edge between two distinct vertices iff they are related in P. An ordered set P is called *chain-complete* iff every nonempty chain has a supremum and an infimum in P. A property of an ordered set is called a *comparability invariant in the class C of ordered sets* iff for all ordered sets $P, Q \in C$ that have isomorphic comparability graphs either both or neither have the property. Recall that an ordered set is said to have the *fixed point property* iff every order-preserving self map has a fixed point. In [2] it has been shown that the fixed point property is a comparability invariant in the class of finite ordered sets.

Here, as well as in other branches of combinatorial theory it is natural to ask how far such a finitary result can be extended to the infinite case. Our main result (Theorem 6.2) is especially interesting in light of the fact that little is known about the fixed point

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property for infinite sets aside from simple generalizations of finite results. It is obvious that the fixed point property is not a comparability invariant on the class of all ordered sets. (Let $\mathbb{N}\cup\{\infty\}$ be the set of natural numbers and the point at infinity with the natural order. Then $\mathbb{N}\cup\{\infty\}$ and $\mathbb{N}$ have the isomorphic comparability graphs but the first set has the fixed point property, while the second does not.) However with restriction to the proper class a comparability invariance result should be possible to be proved. Classes that are often considered as "close" to the class of finite ordered sets are:

1. ordered sets of finite height,
2. ordered sets with no infinite chains,
3. (chain-complete) ordered sets of finite width,
4. (chain-complete) ordered sets with no infinite antichains,
5. chain-complete ordered sets,

as each class shares some important properties with the class of finite ordered sets. For example, it is proved in [7, 8] that each chain-complete ordered set with no infinite antichain is dismantlable to a finite ordered set in finitely many steps. For fixed point theory this makes these ordered sets 'essentially finite' and the fixed point property is a comparability invariant in this class (cf. [9, Theorem 1.7]). The class of chain-complete ordered sets is substantially larger than the class of chain-complete ordered sets with no infinite antichains. For example, it contains the sets with no infinite chains (and thus also the sets of finite height). Yet the class of chain-complete ordered sets shares the following two important properties with the class of finite ordered sets:

1. The Abian–Brown–Pelczar theorem: ([1, 11]) Let P be a chain-complete ordered set and let $f:P\to P$ be order-preserving. If there is a $p\in P$ with $p\sim f(p)$, then f has a fixed point. This is an easy result and a standard tool for finite ordered sets.
2. The existence of a decomposition for chain-complete ordered sets (cf. Lemmas 4.16 and 5.5) that is analogous to the decomposition of finite ordered sets (cf. e.g., [3, Satz 1.2; 6, Theorem 1.2; 10, Theorem 4.2]). The proof of comparability invariance in the finite case in [2] relies heavily on a decomposition result for finite ordered sets (cf. [2, Theorem 1]). Our proof of Theorem 6.2 is clearly inspired by the approach in [2].

Our "$\mathbb{N}\cup\{\infty\}$ vs. $\mathbb{N}$" example shows that there is little hope for comparability invariance of the fixed point property beyond the class of chain-complete ordered sets. i.e., chain-complete sets seem to be the natural habitat for an invariance result regarding the fixed point property and for further investigation of the fixed point property. In fact, the author is unaware of any strong fixed point results for non-chain-complete ordered sets.

The currently starting search for infinitary results in the fixed point theory for ordered sets can also be seen as motivated by the recent resolution of the product problem by Roddy (cf. [12]): If P,Q are finite ordered sets with the fixed point property, then $P\times Q$ also has the fixed point property. Some infinitary generalizations exist (using the above mentioned Li–Milner structure theorem from [8]), but the result is for example unproved for chain-complete sets or for sets of finite height. At present time it is not clear if the product property (P has the product property iff for all ordered sets Q

with the fixed point property we have that $P \times Q$ has the fixed point property) is a comparability invariant on the class of chain-complete ordered sets. If it is not we would have that contrary to the finite case the product property is not equivalent to the fixed point property for chain-complete sets. If it is, this would be another hint that the two properties might be equivalent in general or at least for chain-complete sets.

2. Basic definitions and preliminaries

All results in this paper are stated as generally as possible and chain-completeness is only needed where it is explicitly demanded. We denote $\uparrow p := \{x \in P : x \geqslant p\}$. Recall that an ordered set is called *connected* iff there is a fence $a = f_0 \leqslant f_1 \geqslant f_2 \leqslant \cdots \geqslant b$ between any two elements $a, b \in P$ and that every ordered set with the fixed point property is connected.

The proof of the invariance result in [2] rests on a result ([2, Theorem 1]), which basically states that for finite sets comparability invariance is equivalent to a relatively simple theorem for lexicographic sums. Recall

Definition 2.1. Let T be a nonempty ordered set considered as an index set and let $\{P_t\}_{t \in T}$ be a family of pairwise disjoint nonempty ordered sets that are all disjoint from T. (All lexicographic sum representations in this paper will implicitly be assumed to satisfy this hypothesis. Since we can always achieve this by exchanging some elements if necessary, this is no restriction of generality.) We define the *lexicographic sum* $L\{P_t \mid t \in T\}$ to be $\bigcup_{t \in T} P_t$ ordered by $p_1 \leqslant p_2$ iff
1. $p_i \in P_{t_i}$, $t_1 \neq t_2$ and $t_1 <_T t_2$, or
2. $p_1, p_2 \in P_t$ and $p_1 \leqslant_{P_t} p_2$.

The P_t will be called the *pieces* of the lexicographic sum and T will be called the *index set*. If the index set T is a chain, we will call the lexicographic sum *linear*. A linear lexicographic sum with finitely many pieces $P_1 < P_2 < \cdots < P_n$ is also denoted $P_1 \oplus P_2 \oplus \cdots \oplus P_n$.

Lexicographic sums and their relation to the fixed point property were investigated in [4, 5, 13] for example. In Section 3 we prove a lemma on chain-complete lexicographic sums (cf. Lemma 3.4) that has a similar flavor as the condition in Theorem 1 in [2]. In order to use it we prove in Sections 4 and 5 (cf. Lemmas 4.16 and 5.5) that connected, chain-complete sets can be decomposed into lexicographic sums that are in a sense "comparability invariant" (in the language of [6] all pieces of these lexicographic sums are strongly autonomous).

Definition 2.2. Let P be an ordered set and let $A, B \subseteq P$. Then we will say
1. $A \leqslant B$ iff for all $a \in A$, $b \in B$ we have $a \leqslant b$,
2. $A < B$ iff for all $a \in A$, $b \in B$ we have $a < b$.

When singleton sets are involved we generally omit the set brackets.

Definition 2.3 (cf., e.g., Dreesen, Poguntke and Winkler [2], and Kelly [6]). Let P be an ordered set and $S \subseteq P$. Then $S \neq \emptyset$ is called *order-autonomous* iff for all $p \in P \setminus S$ we have that

1. If there is an $s \in S$ with $p \leqslant s$, then $p \leqslant S$, and
2. If there is an $s \in S$ with $p \geqslant s$, then $p \geqslant S$.

Example 2.4. For every $p \in P$ the set $\{p\}$ is order-autonomous. Moreover, P is order-autonomous in itself. These order-autonomous subsets of an ordered set will be referred to as the *trivial order-autonomous subsets.*

The author's main reference on comparability graphs is Kelly's classical paper [6]. In [6], Kelly develops a decomposition theory for infinite ordered sets, while very clearly pointing out which pitfalls exist in the generalization process. In the language of [6] we show that maximal order-autonomous subsets (cf. Definition 4.8) and minimal chain-links (cf. Definition 5.1) are quasimaximal strongly autonomous sets (cf. Lemmas 4.9 and 5.4). Lemmas 4.12 and 5.3 say that every point in a chain-complete ordered set is contained in a quasimaximal strongly autonomous set, a fact that is not true for sets that are not chain-complete (cf. [6, p. 26]). As a direct proof of our results is only as long as a proof of strong autonomy, etc., of the involved sets we make our approach self-contained and try to use the language of ordered sets exclusively rather than the language of [6]. This is also necessary as the main fixed point lemma (cf. Lemma 3.4) is a theorem on lexicographic sums, not comparability graphs and since chain-completeness is an order-theoretical rather than a graph-theoretical concept. Lemmas 4.16 and 5.5 are providing the canonical decompositions of chain-complete lexicographic sums that are crucial in proving the main result. The above-mentioned lemmas are used in Section 6 to prove that the fixed point property is a comparability invariant for the class of chain-complete ordered sets (cf. Theorem 6.2). Finally, in Section 7, Theorem 7.1 we prove an analogue of Theorem 1 in [2] which is applicable to infinite ordered sets of finite height. The proof of Theorem 7.1 is similar to the proof of Theorem 1 in [2] in that we exhibit a step-by-step transformation from one ordered set to the other without loosing any properties. The proof of Theorem 6.2 is not a transformation between the ordered sets, but similar in spirit, as we go to smaller and smaller sets in the canonical decomposition.

Notation. Whenever we are considering several ordered sets or several orders on the same set, the orders and order-related constructions/properties, etc., will be given an index or a prefix to indicate in which ordered set a certain property holds. No indexing will be used in results and proofs in which only one order is involved.

3. Lemmas on lexicographic sums

Lemma 3.1. *Let $L\{P_t \mid t \in T\}$ be a lexicographic sum with the fixed point property. Then T has the fixed point property.*

Proof. T is a retract of the lexicographic sum. $\square$

Lemma 3.2. *Let T be an ordered set and let $P:=L\{P_t \mid t\in T\}$ be a P-chain-complete lexicographic sum. If the piece P_b contains a chain C that has no supremum in P_b, then $(\uparrow_T b)\setminus\{b\}$ has a T-smallest element s and P_s has a P-smallest element.*

Proof. Let $c:=\bigvee_P C$. Then $c\notin P_t$. Let s be such that $c\in P_s$. Then for all $t>_T b$ and all $p\in P_t$ we have $p\geqslant_P c$. Hence $t\geqslant_T s$ and c is the P-smallest element of P_s. $\square$

Lemma 3.3. *Let T be an ordered set and let $P:=L\{P_t \mid t\in T\}$ be a P-chain-complete lexicographic sum. If $C\subseteq T$ is T-well-ordered, then $\bigvee_T C$ exists. Moreover if $\bigvee_T C\notin C$, then $P_{\bigvee_T C}$ has a P-smallest element.*

Proof. If C is not isomorphic to a limit ordinal the whole statement is trivial, as C has a T-largest element. Otherwise we argue as follows: For every $c\in C$ choose a $p_c\in P_c$. Then $K:=\{p_c : c\in C\}$ is P-well-ordered, as $p_{c_1}\leqslant_P p_{c_2}$ iff $c_1\leqslant_T c_2$. Let $k:=\bigvee_P K$. Then $k\notin P_c$ for all $c\in C$. Let $d\in T$ be such that $k\in P_d$. Then for all $t\in T$ with $t\geqslant_T C$, there is a $p\in P_t$ with $p\geqslant_P K$ and hence $p\geqslant_P k$. Therefore $t\geqslant_T d$ and $d=\bigvee_T C$. Moreover since every element of P_d is a P-upper bound of K, k is the P-smallest element of P_d. $\square$

Lemma 3.4. *Let T be an ordered set and let $P:=L\{P_t \mid t\in T\}$ and $Q:=L\{Q_t \mid t\in T\}$ be chain-complete lexicographic sums. If P has the fixed point property, then for every order-preserving map $f:Q\to Q$ at least one of the following is true:*
1. *f has a fixed point, or*
2. *There is a $t\in T$ such that P_t has the fixed point property, Q_t is chain-complete and f maps Q_t to itself.*

Proof. Let $f:Q\to Q$ be order-preserving. We will construct a countable sequence of sets $\emptyset\neq T^n\subseteq T$, such that
1. For each $n\in\mathbb{N}$ there is a $t_n\in T^n\setminus T^{n+1}$ with $T^{n+1}=(\downarrow_{T^n} t_n)\setminus\{t_n\}$ or $T^{n+1}=(\uparrow_{T^n} t_n)\setminus\{t_n\}$,
2. $Q^n:=L\{Q_t \mid t\in T^n\}$ is chain-complete and f maps Q^n to itself,
3. $P^n:=L\{P_t \mid t\in T^n\}$ has the fixed point property. In particular, T^n has the fixed point property.

To start the induction we set $T^0:=T$. For the inductive step assume that T^n and t_{n-1} have already been chosen. Consider the mapping

$$F:T^n\to(\mathcal{P}(T^n)\setminus\{\emptyset\});\qquad t\mapsto\{s\in T^n : (\exists q\in Q_t) f(q)\in Q_s\}.$$

There must be a $t_b\in T^n$ with $F(t_b)=\{t_b\}$, as otherwise one could choose $g(t)\in F(t)\setminus\{t\}$ for every $t\in T^n$ and g would be a fixed point free T^n-order-preserving self map of T^n, contradicting the inductive assumption 3. Thus f maps Q_{t_b} to itself, which implies f maps the linear lexicographic sum

$$G := L\{Q_t \mid t \in T^n,\ t <_{T^n} t_b\} \oplus Q_{t_b} \oplus L\{Q_t \mid t \in T^n,\ t >_{T^n} t_b\}$$

to itself. If Q_{t_b} is not chain-complete, then by Lemma 3.2 $L\{Q_t \mid t \in T^n, t <_{T^n} t_b\}$ has a largest element or $L\{Q_t \mid t \in T^n, t >_{T^n} t_b\}$ has a smallest element. Call this element x. Then x is Q-related to each point in G and since f maps G to itself $f(x) \sim_Q x$ and f has a fixed point. In this case we stop. Thus, in the following, we can assume Q_{t_b} is chain-complete. If P_{t_b} has the fixed point property, we stop. If Q_{t_b} has a Q-largest or a Q-smallest element q, then $f(q)$ is Q-comparable to q, f has a fixed point and we stop. If none of the above is the case, then P_{t_b} does not have the fixed point property and there must be

1. a $q \in L\{Q_t \mid t \in T^n,\ t <_{T^n} t_b\}$ with $f(q) \notin L\{Q_t \mid t \in T^n,\ t <_{T^n} t_b\}$, in which case f has a fixed point and we stop, or
2. a $q \in L\{Q_t \mid t \in T^n,\ t >_{T^n} t_b\}$ with $f(q) \notin L\{Q_t \mid t \in T^n,\ t >_{T^n} t_b\}$, in which case f has a fixed point and we stop, or
3. f maps Q_{t_b}, $L\{Q_t \mid t \in T^n,\ t >_{T^n} t_b\}$ and $L\{Q_t \mid t \in T^n,\ t <_{T^n} t_b\}$ to themselves and at least one of $L\{P_t \mid t \in T^n,\ t >_{T^n} t_b\}$ and $L\{P_t \mid t \in T^n,\ t <_{T^n} t_b\}$ has the fixed point property (cf [13, Corollary 4.2]).

Since in the first two cases we are done we only need to consider 3. Suppose, without loss of generality, that $L\{P_t \mid t \in T^n,\ t <_{T^n} t_b\}$ has the fixed point property. Applying Lemma 3.2 to G, we infer, since Q_{t_b} was not supposed to have a Q-smallest element, that $L\{Q_t \mid t \in T^n,\ t <_{T^n} t_b\}$ is chain-complete. We let $T^{n+1} := \{t \in T^n : t <_{T^n} t_b\}$, $t_n := t_b$ and continue the induction. (It is obvious that T^{n+1}, t_n satisfy inductive assumptions 1–3.)

If the above induction stops at a finite $n \in \mathbb{N}$, then we are done. If not, the set $C := \{t_n : n \in \mathbb{N}\}$ is a T-chain, which is the disjoint union of $C_l := \{t_n \in C : t_n <_T T^{n+1}\}$ and $C_u := \{t_n \in C : t_n >_T T^{n+1}\}$. Moreover $C_l <_T C_u$. Also, for $n < m$ with t_n, $t_m \in C_l$ we have $T^n \supset T^m$ and hence $t_n <_T t_m$, and for $n < m$ with $t_n, t_m \in C_u$ we have $T^n \supset T^m$ and hence $t_n >_T t_m$. Without loss of generality, we can assume that C_l is infinite and hence isomorphic to the natural numbers. First suppose that C_u has a minimum $t_m \in T$ and that $t_m = \bigvee_T C_l$. Then by Lemma 3.3, Q_{t_m} has a Q-smallest element, which cannot be, as the induction would have stopped at m. Thus the set $T^\infty := \{t \in T : C_l <_T t <_T C_u\} = \bigcap_{n \in \mathbb{N}} T^n \neq \emptyset$. Moreover by Lemma 3.3 the set $Q^\infty := L\{Q_t \mid t \in T^\infty\} = \bigcap_{n \in \mathbb{N}} L\{Q_t \mid t \in T^n\}$ has a Q-smallest element q_s. Let $q \in Q^\infty$. For each $n \in \mathbb{N}$ we have $q \in L\{Q_t \mid t \in T^n\}$ and hence $f(q) \in L\{Q_t \mid t \in T^n\}$. Thus f maps Q^∞ to itself and hence $f(q_s) \geqslant_Q q_s$ and f has a fixed point. $\square$

4. Lemmas on order-autonomous subsets

In general, it is not true that if P, Q have the same comparability graph and $S \subseteq P$ is order-autonomous as a subset of P, then S is also order-autonomous as a subset of Q

([6, bottom of p. 13]). Lemmas 4.6 and 4.16 are vehicles that allow the transplantation of order-autonomous sets.

Lemma 4.1. *Let P be an ordered set and let $S \subseteq P$ be a disconnected order-autonomous subset. Then every component C of S is order-autonomous in P.*

Proof. Let C be a component of the disconnected order-autonomous subset S of P and let $p \in P \setminus C$ with $p \geqslant c$ for some $c \in C$. Then $p \notin S$, hence $p > S$ and in particular $p > C$. The other comparability is handled dually. □

Definition 4.2. Let P be an ordered set and let $a, b \in P$ with $a \leqslant b$. We define

$$[a,b] := \{p \in P \colon a \leqslant p \leqslant b\}.$$

We call $C \subseteq P$ *convex* iff for all $a, b \in C$ with $a \leqslant b$ we have $[a,b] \subseteq C$. (Note that this need not imply that C is connected.) For $D \subseteq P$ we define the *convex hull of* D to be

$$\mathrm{con}(D) := \bigcup \{[a,b] \colon a, b \in D;\ a \leqslant b\}.$$

Proposition 4.3. *Let P be an ordered set and let $D \subseteq P$. Then,* con(D) *is contained in every convex superset of D (trivial). Moreover every $d \in D$ is contained in a maximal (with respect to inclusion) P-convex subset of D. (Zorn's Lemma).* □

Proposition 4.4 (also cf. Kelly [6], remarks on p. 13). *Let P be an ordered set and let $\emptyset \neq S \subseteq P$ be order-autonomous. Then S is convex.*

Proof. Let $a, b \in S$ with $a \leqslant b$. Assume there is a $p \in P \setminus S$ with $a < p < b$. Then $p > S$ and $p < S$, i.e., in particular $p > a$ and $p < a$, which is impossible. □

Lemma 4.5. *If P is connected and $S \subset P$ is an order-autonomous, proper subset of P, then there is an upper or a lower bound of S in $P \setminus S$.*

Proof. Let $S \subseteq P$ be an order-autonomous subset with $P \setminus S \neq \emptyset$. Since P is connected, there is a $p \in P \setminus S$ that is related to an $s \in S$, say $p > s$. Then by order-autonomy of S we have $p > S$. □

Lemma 4.6. *Let $P = (U, \leqslant_P)$ and $Q = (U, \leqslant_Q)$ be two ordered sets with the same underlying set U and the same comparability graph. If $S \subseteq P$ is P-order-autonomous, then $\mathrm{con}_Q(S)$ is Q-order-autonomous. Moreover, every maximal (with respect to inclusion) Q-convex subset of S is Q-order-autonomous.*

Proof. Let $x \in Q \setminus \mathrm{con}_Q(S)$ be such that there is an $s \in \mathrm{con}_Q(S)$ with $x >_Q s$. Then there is an $\tilde{s} \in S$ with $x >_Q s \geqslant_Q \tilde{s}$. Hence $x \sim_Q \tilde{s}$, which means $x \sim_P \tilde{s}$. Therefore,

$x \sim_P t$ for all $t \in S$, which in turn implies $x \sim_Q t$ for all $t \in S$. As $x >_Q \tilde{s}$ and $x \notin \mathrm{con}_Q(S)$, there is no element of S that is $\geqslant_Q x$. Therefore $x >_Q t$ for all $t \in S$, i.e., $x >_Q S$, which implies $x >_Q \mathrm{con}_Q(S)$. The case $x <_Q s \in \mathrm{con}_Q(S)$ is treated dually.

To prove the 'moreover'-part let $C \subseteq S$ be a maximal Q-convex subset of S. Let $x \in Q \setminus C$ be such that $x >_Q c$ for some $c \in C$. (The dual case is similar.) If $x \notin S$, then (similar to the above) x is related to all elements of S. Thus x is related to all elements of C and since C is convex, $x >_Q C$. If $x \in S$, then there is a $b \in C$ and a $q \notin S$ such that $x >_Q q >_Q b$ (otherwise $\mathrm{con}_Q(C \cup \{x\})$ would be a larger Q-convex subset of S). As above $q >_Q C$ and thus $x >_Q C$. □

Lemma 4.7. *Let $P = (U, \leqslant_P)$ and $Q = (U, \leqslant_Q)$ be two ordered sets with the same underlying set U and the same comparability graph. If $S \subset P$ is a proper P-order-autonomous subset of P and $\mathrm{con}_Q(S) = Q$, then Q has a nontrivial decomposition as a linear lexicographic sum.*

Proof. Let $H \subset S$ be a maximal Q-convex subset of S. Then $\emptyset \neq H \neq Q$ and by Lemma 4.6, H is Q-order-autonomous. Every element of $Q \setminus S$ is Q-related to some element (and hence all elements) of S as $\mathrm{con}_Q(S) = Q$ and S is P-order-autonomous. Hence every element of $Q \setminus S$ is Q-related to all elements of H. If $x \in S \setminus H$, then there is an $h \in H$ with $x \sim_Q h$ (otherwise $\mathrm{con}_Q(H \cup \{x\})$ is a larger Q-convex subset of S), say $x >_Q h$ (one treats the other case dually). Thus there is a $q \in Q \setminus S$ and a $b \in H$ with $x >_Q q >_Q b$ (otherwise $\mathrm{con}_Q(H \cup \{x\})$ is a larger Q-convex subset of S). Thus q is related to all elements of H and since H is convex $q >_Q H$. Thus $x >_Q H$. Hence all elements of $Q \setminus H$ are Q-related to all elements of H and $Q = \{q \in Q: q <_Q H\} \oplus H \oplus \{q \in Q: q >_Q H\}$, where at least two summands are nonempty. □

Definition 4.8. An order-autonomous subset S of the ordered set P is called *maximal* iff $S \neq P$ and for all order-autonomous subsets $\tilde{S} \subseteq P$ with $S \subseteq \tilde{S}$ we have $\tilde{S} \in \{S, P\}$.

Lemma 4.9. *Let $P = (U, \leqslant_P)$ and $Q = (U, \leqslant_Q)$ be two ordered sets with the same underlying set U and the same comparability graph. Assume that P, Q have no nontrivial decomposition as linear lexicographic sums. If $S \subset P$ is a maximal P-order-autonomous subset of P, then S is also a maximal Q-order-autonomous subset of Q.*

Proof. By Lemmas 4.6 and 4.7, $\mathrm{con}_Q(S)$ is Q-order-autonomous and not equal to Q. Repetition of this argument shows that $\mathrm{con}_P(\mathrm{con}_Q(S))$ is P-order-autonomous and not equal to P. Since the underlying sets satisfy

$$S \subseteq \mathrm{con}_Q(S) \subseteq \mathrm{con}_P(\mathrm{con}_Q(S)) \subseteq P$$

and S was a maximal P-order-autonomous subset of P, we infer that except P all sets above must be equal. Thus $S = \mathrm{con}_Q(S)$ is Q-order-autonomous. Now let $B \subset Q$ be a proper Q-order-autonomous subset of Q such that $S \subseteq B$. Then $S \subseteq B \subseteq \mathrm{con}_P(B) \neq P$, and $\mathrm{con}_P(B)$ is P-order-autonomous. Maximal P-order-autonomy of S shows $S = B = \mathrm{con}_P(B)$ and thus S is also maximal Q-order-autonomous. $\square$

Lemma 4.10. *Let $\{S_\alpha\}_{\alpha \in I}$ be a family of order-autonomous subsets of the ordered set P with $\bigcap_{\alpha\in I} S_\alpha \neq \emptyset$. Then $\bigcup_{\alpha\in I} S_\alpha$ is order-autonomous.*

Proof. Let $s \in \bigcap_{\alpha\in I} S_\alpha$. Suppose $p \in P \setminus \bigcup_{\alpha\in I} S_\alpha$ satisfies $p < q \in \bigcup_{\alpha\in I} S_\alpha$. Then $p < s$ and hence $p < S_\alpha$ for all $\alpha \in I$. The dual situation is treated similarly. $\square$

Lemma 4.11. *Let P be an ordered set and suppose that S_1 and S_2 are connected order-autonomous subsets of P with $S_1 \cap S_2 \neq \emptyset$. Then we have*
1. *$S_1 \subseteq S_2$, or*
2. *$S_2 \subseteq S_1$, or*
3. *$S_1 \cup S_2$ is the linear lexicographic sum $(S_2 \setminus S_1) \oplus (S_1 \cap S_2) \oplus (S_1 \setminus S_2)$ with all pieces nonempty, or*
4. *$S_1 \cup S_2$ is the linear lexicographic sum $(S_1 \setminus S_2) \oplus (S_1 \cap S_2) \oplus (S_2 \setminus S_1)$ with all pieces nonempty.*

Proof. Assume that $S_1 \setminus S_2 \neq \emptyset$ and $S_2 \setminus S_1 \neq \emptyset$. By Lemma 4.10 we have that $S_1 \cup S_2$ is order-autonomous. Since $S_1 \cup S_2$ is connected there is a $p_0 \in S_2 \setminus S_1$ such that p_0 is related to an element of S_1. Hence p_0 is an upper or a lower bound of S_1, say without loss of generality an upper bound (this is case 4, case 3 is treated similarly). Hence every $x \in S_1 \setminus S_2$ is below p_0, i.e., below S_2. Thus $S_1 \cup S_2 = (S_1 \setminus S_2) \oplus S_2$. Pick $x_0 \in S_1 \setminus S_2$. Then every $p \in S_2 \setminus S_1$ is above x_0 and hence above S_1. Thus $S_1 \cup S_2 = S_1 \oplus (S_2 \setminus S_1)$, which implies the claim. $\square$

Lemma 4.12. *Let P be a connected, chain-complete ordered set that has no nontrivial representation as a linear lexicographic sum. Then every connected order-autonomous proper subset $O \subset P$ is contained in a unique maximal connected order-autonomous proper subset of P.*

Proof. Let $S \subset P$ be a connected order-autonomous proper subset of P. Then the set $\mathcal{S}$ of all connected order-autonomous supersets C of S such that each point in $C \setminus S$ is related to all points in S is not empty ($S \in \mathcal{S}$). Let $V(S) := \bigcup \mathcal{S}$. Clearly, $V(S)$ is connected and (by Lemma 4.10) order-autonomous. Since P has no nontrivial representation as a linear lexicographic sum $V(S) \neq P$.

For $O \subset P$ a connected order-autonomous proper subset of P, let

$$\mathcal{O} := \{V(U) : P \neq U \supseteq O \text{ and } U \text{ is connected and order-autonomous}\}.$$

Since $P \neq V(S)$ for any proper connected order-autonomous subset we have $P \notin \mathcal{O}$. Moreover, by definition of $V(U)$ and Lemma 4.11 $\mathcal{O}$ is a chain with respect to inclusion and $H := \bigcup \mathcal{O}$ is connected and by Lemma 4.10 order-autonomous. If $H \in \mathcal{O}$, we are done, as $H \neq P$ must be a maximal connected order-autonomous proper subset of P

that contains O. Assume $H \notin \mathcal{O}$. Then $\mathcal{O}$ has no largest element. Since by Lemma 4.5 for each element $C \in \mathcal{O}$ there is a $p \in P \setminus C$ with $p > C$ or $p < C$, we can find an infinite cofinal well-ordered subchain $\mathcal{D} := \{D_\alpha : \alpha \in \Gamma\} \subseteq \mathcal{O}$ such that for each α there is a $d_\alpha \in D_{\alpha+1} \setminus D_\alpha$, that is an upper or a lower bound of D_α. Without loss of generality, we can assume that there is a cofinal subchain $\{D_{\alpha_\beta} : \beta \in \Gamma\}$ of $\{D_\alpha : \alpha \in \Gamma\}$ such that d_{α_β} is an upper bound of D_{α_β} for all $\beta \in \Gamma$. (The d_{α_β} are well-ordered.) Let $d := \bigvee_{\beta \in \Gamma} d_{\alpha_\beta}$. Then $d \notin D_{\alpha_\beta}$ for all $\beta \in \Gamma$ and hence, since $\{D_{\alpha_\beta} : \beta \in \Gamma\}$ was cofinal, $d \notin D_\alpha$ for all $\alpha \in \Gamma$. This implies that d is an upper bound of H with $d \notin H$. Thus $H \neq P$, which implies $P \neq V(H) \in \mathcal{O}$. As $V(H) \supseteq H$, this implies $\mathcal{O} \ni V(H) = H$, which is a contradiction. Hence $H \neq P$ is a maximal connected order-autonomous superset of O.

Uniqueness follows from Lemma 4.11: Suppose $O \subseteq H_1, H_2$ with H_1, H_2 being distinct maximal connected order-autonomous subsets. Then $H_1 \cup H_2$ is connected and order-autonomous and hence equal to P. Now Lemma 4.11 leads to a contradiction. □

Lemma 4.13. *Let P be a connected, chain-complete ordered set that has no nontrivial representation as a linear lexicographic sum. Then every connected order-autonomous subset $S \subseteq P$ is contained in a maximal order-autonomous subset.*

Proof. Let H be the maximal connected order-autonomous subset that contains S. Let $\mathcal{L}$ be the set of all order-autonomous subsets of P that have the same strict upper and lower bounds as H and let $K := \bigcup \mathcal{L}$. Since $H \neq P$ and P is connected we have $K \neq P$. If $x \in P \setminus K$ satisfies $x > k$ for some $k \in K$, then $x > H$ and hence $x > K$ (other comparability dually), and hence K is order-autonomous. Suppose $M \supset K$ is an order-autonomous superset of K. Then there is an $m \in M \setminus K$ that is comparable to an element of H (otherwise $M \setminus K$ is a component of M and hence in $\mathcal{L}$, contradiction). Thus K is a proper subset of a component of M. Since H was maximal connected order-autonomous this implies that the component of M that contains K is P and hence K is a maximal order-autonomous subset of P. □

Lemma 4.14. *Let P be a connected, chain-complete ordered set that has no nontrivial representation as a linear lexicographic sum. Then every $p \in P$ is contained in a unique maximal order-autonomous subset of P.*

Proof. Existence is provided by Lemma 4.13. To prove uniqueness, assume S_1 and S_2 are two distinct maximal order-autonomous subsets of P such that $S_1 \cap S_2 \neq \emptyset$. Then $S_1 \cup S_2 = P$. Since P is connected there is a point $s_1 \in S_1 \setminus S_2$ that is related to a point $s_2 \in S_2$. Similar to the proof of Lemma 4.11 we now construct a linear lexicographic sum decomposition, which is a contradiction. □

Remark 4.15. Note that in general maximal order-autonomous subsets need not be chain-complete, even when P is chain-complete.

Lemma 4.16. *Let P be a connected, chain-complete ordered set that has no nontrivial representation as a linear lexicographic sum. Then*

$$\mathcal{T} := \{S \subseteq P\colon S \text{ is a maximal order-autonomous subset}\}$$

is ordered via $(\leqslant_{\mathcal{T},P}) := (\leqslant_P \cup =)$, where $\leqslant_P$ is induced by the order on P as in Definition 2.2 and $=$ is equality of sets. Moreover,
1. *P can be represented as the lexicographic sum $L\{S \mid S \in \mathcal{T}\}$,*
2. *$\mathcal{T}$ is chain-complete,*
3. *all order-autonomous subsets of $\mathcal{T}$ are singletons or equal to $\mathcal{T}$,*
4. *if Q is another chain-complete ordered set with the same underlying set U and the same comparability graph, then $Q = L_Q\{S \mid S \in \mathcal{T}\}$, where $\mathcal{T}$ is ordered by $\leqslant_Q \cup =$ with $\leqslant_Q$ as induced by the order in Q as in Definition 2.2, $=$ is equality of sets and the pieces are ordered subsets of Q. 2 and 3 also hold for the orders induced by Q on the respective sets.*

Proof. All elements of $\mathcal{T}$ are maximal P-order-autonomous subsets of P. Thus by Lemma 4.14 all elements of $\mathcal{T}$ are pairwise disjoint. Now it is clear that $(\mathcal{T}, \leqslant_{\mathcal{T},P})$ is an ordered set. To see 1, note that the underlying sets for P and $L\{S \mid S \in \mathcal{T}\}$ are the same. Consider $p_1, p_2 \in P$ and let S_i be the maximal order-autonomous subset containing p_i. Then $p_1 \leqslant_P p_2$ iff ($S_1 = S_2$ and $p_1 \leqslant_{S_1} p_2$) or $S_1 <_{\mathcal{T},P} S_2$. 2 follows since $\mathcal{T}$ is isomorphic to a retract of P and hence chain-complete. For 3 suppose that $\mathcal{T}$ has a $\mathcal{T}$-order-autonomous subset $\mathcal{B} \neq \mathcal{T}$ that is not a singleton. Then $L\{S \mid S \in \mathcal{B}\} \subseteq P$ is a P-order-autonomous proper subset of P and a proper superset of all the maximal P-order-autonomous subsets $S \in \mathcal{B}$, which is a contradiction. Hence all $\mathcal{T}$-order-autonomous subsets of $\mathcal{T}$ are singletons or equal to $\mathcal{T}$. To see 4 recall that by Lemma 4.9 P and Q have the same maximal order-autonomous subsets. 2 and 3 can now be proved in Q just as they were proved in P above. □

5. Lemmas on order-autonomous subsets in linear lexicographic sums

The results of the last section leave us with the task find an analogue of Lemma 4.16 for linear lexicographic sums. This is possible by introducing the notion of a minimal chain-link.

Definition 5.1. An order-autonomous proper subset $S \neq P$ of the ordered set P is called
1. A *chain-link* iff every element of S is related to all elements in $P \setminus S$,
2. A *minimal chain-link* iff S is a chain-link and every proper order-autonomous subset of S is not a chain-link.

Lemma 5.2. *Let P be an ordered set. Then P contains a chain-link iff P has a nontrivial representation as a linear lexicographic sum.*

Proof. To prove '$\Rightarrow$' let $C \subseteq P$ be a chain-link. Then $\{p \in P\colon p < C\} \neq \emptyset$ or $\{p \in P\colon p > C\} \neq \emptyset$, and P is the linear lexicographic sum $\{p \in P\colon p < C\} \oplus C \oplus \{p \in P\colon p > C\}$. For '$\Leftarrow$' notice that any piece in a nontrivial linear lexicographic sum representation of P is a chain-link. □

Lemma 5.3. *Let P be an ordered set that contains a chain-link. Then every element $p \in P$ is contained in a unique minimal chain-link.*

Proof. Let $p \in P$. Since P is a linear lexicographic sum and every piece of such a sum is a chain-link, p is contained in a chain-link. Let

$$\mathcal{A} := \{L \subseteq P\colon L \text{ is a chain-link and } p \in L\}.$$

Let $I := \bigcap \mathcal{A} \neq \emptyset$ ($p \in I$!). Now let $x \in I$. Then $x \in A$ for all $A \in \mathcal{A}$. Hence x is related to all elements in $\bigcup_{A \in \mathcal{A}} P \setminus A = P \setminus \bigcap_{A \in \mathcal{A}} A = P \setminus I$. Thus I is a chain-link that contains p. If I would contain another chain-link, then I could be represented as the linear lexicographic sum $L \oplus U$ of the two chain-links L and U. By definition of I neither L nor U could contain p, a contradiction. Hence I is a minimal chain-link. As every chain-link that contains p must contain I, I is the unique minimal chain-link that contains p. □

Lemma 5.4. *Let $P = (U, \leqslant_P)$ and $Q = (U, \leqslant_Q)$ be two ordered sets with the same underlying set U and the same comparability graph. Suppose that P has a nontrivial representation as a linear lexicographic sum. Then every minimal P-chain-link is also a minimal Q-chain-link. In particular, Q has a nontrivial representation as a linear lexicographic sum.*

Proof. Let $S \subseteq U$ be a minimal P-chain-link. Let H be a maximal Q-convex subset of S. Then by Lemma 4.6 H is Q-order-autonomous. Using the exact same argument as in the proof of Lemma 4.7 we infer that H is a Q-chain-link. Let $B_Q(S) \subseteq H \subseteq S$ be a minimal Q-chain-link. Repeating this procedure with $B_Q(S)$ in the ordered set P, we obtain a minimal P-chain-link $B_P(B_Q(S))$ such that $S \supseteq B_Q(S) \supseteq B_P(B_Q(S))$. By minimality of S we conclude that all the above sets must be equal, which implies that $S = H$ is a minimal Q-chain-link. □

Lemma 5.5. *Let P be an ordered set that contains a chain-link. The set*

$$\mathcal{C} := \{C \subseteq P\colon C \text{ is a minimal } P\text{-chain-link}\}$$

is totally ordered by $(\leqslant_{\mathcal{C},P}) := (\leqslant_P \cup =)$, where $\leqslant_P$ is induced by the order on P as in Definition 2.2 and $=$ is equality of sets. Moreover,

1. *P can be represented as the linear lexicographic sum $L\{S \mid S \in \mathcal{C}\}$,*
2. *If P is chain-complete, then so is $\mathcal{C}$,*
3. *If Q is another ordered set with the same underlying set U and the same comparability graph, then $Q = L_Q\{S \mid S \in \mathcal{C}\}$, where $\mathcal{C}$ is ordered by $\leqslant_Q \cup =$ with $\leqslant_Q$*

as induced by the order in Q via Definition 2.2 and the pieces are ordered subsets of Q. All the pieces are minimal Q-chain-links. If Q is chain-complete, then so is $\mathcal{C}$ with the order induced by Q.

Finally, if $\mathcal{C}$ is infinite and P is chain-complete, then P contains an element s that is P-related to all other elements.

Proof. Let $E_1 \neq E_2$ be elements of $\mathcal{C}$. Then every element of $P \setminus E_1$ is P-related to all elements of E_1. Suppose, without loss of generality, there is an $x \in E_2$ with $x \geqslant_P E_1$. Then, since E_2 is order-autonomous, $E_2 \geqslant_P p$ for all $p \in E_1$, i.e., $E_2 \geqslant_P E_1$. Thus $\mathcal{C}$ is a chain. To prove 1 note that the underlying set of $L\{S \mid S \in \mathcal{C}\}$ is P and for $p_1, p_2 \in P$ that are contained in the minimal P-chain-links E_1, E_2 respectively we have $p_1 \leqslant_P p_2$ iff $E_1 <_{\mathcal{C},P} E_2$ or $E_1 = E_2$ and $p_1 \leqslant_{E_1} p_2$. Since $\mathcal{C}$ is thus isomorphic to a retract of P, we infer that $\mathcal{C}$ is chain-complete if P is, i.e., 2. To prove 3 note that by Lemma 5.4, P and Q have the same minimal chain-links. Thus we can prove as above that $Q = L_Q\{S \mid S \in \mathcal{C}\}$ with all orders induced by Q, and that if Q is chain-complete, $\mathcal{C}$ is chain-complete with respect to the order induced by Q. Finally if $\mathcal{C}$ is infinite and P is chain-complete, then there is without loss of generality, a $\mathcal{C}$-well-ordered chain $\mathcal{W} \subseteq \mathcal{C}$. Let $S := \bigvee_{\mathcal{C}} \mathcal{W}$. Then by Lemma 3.3, S has a P-smallest element. Since S was a P-chain-link, s is P-related to every element of S and of $P \setminus S$. $\square$

6. The main result

We need the following lemma that is a consequence of the work in [6]. This is the only place at which the paper is not completely self-contained. An ordered set with $\geqslant 2$ points and no nontrivial order-autonomous subsets (i.e., such that all order-autonomous subsets are singletons or the whole set) is called prime. By the remarks in [6, p.24] a prime ordered set has a prime comparability graph (the comparability graph has only singletons or the whole vertex set as autonomous subgraphs). By Corollary 4.4 in [6] each prime comparability graph is uniquely orderable (i.e., there are only two possible orders that induce this graph and these orders are duals of each other). Hence

Lemma 6.1 (cf. [6], Corollary 4.4 and Definition and Remarks on p. 24). *Let P, Q be ordered sets with no nontrivial order-autonomous subsets and isomorphic comparability graphs. Then P is isomorphic to Q or the dual of Q.* $\square$

Theorem 6.2. *Let P, Q be chain-complete ordered sets with isomorphic comparability graphs. Then P has the fixed point property iff Q has the fixed point property.*

Proof. After possibly exchanging some points, we can assume that $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ have the same underlying set $U := P = Q$ and that $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ have the same comparability graph G. If either of $(P, \leqslant_P)$, $(Q, \leqslant_Q)$ is not connected, then G is not connected and hence the other ordered set also is not connected. Hence in this case there is nothing to prove. If – say – $(P, \leqslant_P)$ contains an element s that is P-related to all other elements in P, then s is also Q-related to all other elements in Q, and again there is nothing to prove. In the following we can thus assume that $(P, \leqslant_P)$, $(Q, \leqslant_Q)$ and G are connected and that neither P nor Q contains an element that is related to all other elements of P resp. Q. Also it is clear that we only need to prove one direction, so we will assume that $(P, \leqslant_P)$ has the fixed point property. Let $f:(Q, \leqslant_Q) \to (Q, \leqslant_Q)$ be an order-preserving map. We will show that f has a fixed point or there is a subset $S' \subseteq U$ such that

1. f maps S' to itself,
2. S' is chain-complete as an ordered subset of P and as an ordered subset of Q,
3. S' has the fixed point property when regarded as an ordered subset of P,
4. S' has a strict Q-upper bound or Q-lower bound.

First suppose P contains a P-chain-link. Let $L_P\{S_P \mid S_P \in \mathcal{C}_P\}$ be a representation of P as a lexicographic sum as in Lemma 5.5, where the index 'P' signalizes that all sets carry the order induced by P. Since P does not contain any point that is P-related to all points in P, the set $\mathcal{C}$ is finite. By part 3 of Theorem 5.5, Q has a representation $L_Q\{S_Q \mid S_Q \in \mathcal{C}_Q\}$, the index Q signalizing that the sets carry the order induced by Q. (For the underlying sets we have $\mathcal{C} = \mathcal{C}_P = \mathcal{C}_Q$ and $S = S_P = S_Q$.) By Lemma 3.2 and the assumption that neither P nor Q contains a point that is related to all others, every S_P, S_Q is chain-complete in P resp. Q. If there is an S_Q that f does not map to itself, then f has a fixed point and we are done. Otherwise, since there must be an S_P that has the fixed point property (cf. [13, Corollary 4.2]), there is a set $S' \in \mathcal{C}$ such that S'_P, S'_Q are chain-complete, f maps S'_Q to itself and S'_P has the fixed point property. Clearly, S'_P and S'_Q have the same comparability graph, hence S'_Q is connected.

If P contains no P-chain-link we represent P as a lexicographic sum $L_P\{S_P \mid S_P \in \mathcal{T}_P\}$ as in Lemma 4.16, the index 'P' indicating that $\mathcal{T}_P$ and the pieces S_P carry the order induced by P. Let $\mathcal{T}$ be the underlying set of $\mathcal{T}_P$. Then no element of $\mathcal{T}_P$ is $\mathcal{T}_P$-related to all other elements of $\mathcal{T}_P$, since otherwise P has a nontrivial representation as a linear lexicographic sum. By part 4 of Lemma 4.16 $Q = L_Q\{S_Q \mid S_Q \in \mathcal{T}_Q\}$, with the index 'Q' signalizing that all sets inherited their order from Q. Since P and Q have the same comparability graphs, $\mathcal{T}_P$ and $\mathcal{T}_Q$ have the same comparability graphs. By Lemma 4.16 part 3 neither $\mathcal{T}_P$, nor $\mathcal{T}_Q$ contains a nontrivial order-autonomous subset. Hence by Lemma 6.1 we have $\mathcal{T}_P = \mathcal{T}_Q$ or $\mathcal{T}_P = \mathcal{T}_Q^d$, the dual of $\mathcal{T}_Q$. If necessary on reversing the order on Q (and thus on $\mathcal{T}_Q$) we infer via Lemma 3.4 that f has a fixed point, or there is a maximal chain-complete order-autonomous subset S'_Q of Q such that f maps S'_Q to itself and S'_P has the fixed point property. Since S'_P has the fixed point property, every nonempty chain in S'_P has a P-upper and a P-lower bound in S'_P and hence a supremum and an infimum in S'_P. Thus S'_P is chain-complete. Clearly S'_P and S'_Q have the same comparability graph and are hence in particular both connected.

In this fashion we can inductively construct a sequence of pairs of connected chain-complete ordered sets (P_n, Q_n) such that

1. Each P_n is a lexicographic sum whose pieces are pieces of P and each Q_n is a lexicographic sum whose pieces are pieces of Q,
2. f maps Q_n to itself,
3. P_n has the fixed point property,
4. P_n and Q_n have the same comparability graph,
5. There is a Q-upper or a Q-lower bound b_n of Q_{n+1} in $Q_n \setminus Q_{n+1}$.

All that is needed to do is to let $P_0 := P$, $Q_0 := Q$ and repeat the initial argument with (P_n, Q_n) instead of (P, Q) until a fixed point of f is found. If no fixed point is found in a step and if no point in $S'_{P_n}(S'_{Q_n})$ is related to all points in $S'_{P_n}(S'_{Q_n})$ (in which case f has a fixed point and we are done), we let $(P_{n+1}, Q_{n+1}) := (S'_{P_n}, S'_{Q_n})$. If no fixed point is found after finitely many steps, we consider $Q_\infty := \bigcap_{n\in\mathbb{N}} Q_n$. Similar to the argument at the end of the proof of Lemma 3.4 we prove that Q_∞ is nonempty and has a Q-largest or a Q-smallest element. Clearly f maps Q_∞ to itself. Thus also in this case f has a fixed point. □

7. An analogue of theorem 1 in [2]

Theorem 1 in [2] shows that for finite sets comparability invariance is equivalent to a simple result about lexicographic sums. The value of such a result is that any parameter that can be explicitly computed for lexicographic sums from data on the pieces and the index set can quickly be proved to be a comparability invariant. The authors then state that no useful analogue of their result was found for larger classes of ordered sets. Theorem 7.1 is such an analogue for the larger class of ordered sets of finite height. The proof of Theorem 6.2 shows that it should be hard to find a generalization for chain-complete sets, as for these sets comparability invariance is equivalent to a result on lexicographic sums plus 'some way to push past the limit ordinal'. In non-chain-complete sets one would even need to take infinite nestings such as the chain $\mathcal{O}$ in the proof of Lemma 4.12 into account. (Their union could be the whole set if P is not chain-complete.) It seems possible to write down such conditions, but their value currently seems doubtful to the author.

If we wish to consider a property as a parameter $\alpha(\cdot)$, then $\mathcal{C}$-comparability invariance translates into $\alpha(P) = \alpha(Q)$ for all $P, Q \in \mathcal{C}$ with isomorphic comparability graphs. This allows a compact way to state the following theorem.

Theorem 7.1. *Let α be a parameter of ordered sets such that for any ordered sets $L\{P_t \mid t \in T\}$ of finite height, we have that*

$$\alpha(L\{P_t \mid t \in T\}) = \alpha(L\{\tilde{P}_t \mid t \in T\}),$$

where $\tilde{P}_t \in \{P_t, P_t^d\}$. Then α is a comparability invariant on the class of all ordered sets of finite height.

Proof. Let $P = (U, \leqslant_P)$ and $Q = (U, \leqslant_Q)$ be ordered sets with the same comparability graph and let α be a parameter as above. We will construct a sequence of lexicographic sums $\{L\{S \mid S \in T_k\}\}_{k=1,\dots,n}$ such that $P = L\{S \mid S \in T_1\}$, $Q = L\{S \mid S \in T_n\}$ and for $k = 1, \dots, n-1$ we have $\alpha(L\{S \mid S \in T_k\}) = \alpha(L\{S \mid S \in T_{k+1}\})$. To do so let $T_1 := \{P\}$. In the inductive step assume that

1. $\bigcup T_k = U$,
2. Every $S \in T_k$ is order-autonomous in P and in Q and has the same comparability graph as a subset of P as when regarded as a subset of Q,
3. Every $S \in T_k$ carries the order induced on it by P or its dual,
4. T_k carries the order $\leqslant_k := \leqslant_Q \bigcup =$ induced on it by Q, where $\leqslant_Q$ is as in Definition 2.2 and $=$ is equality of sets,
5. $\alpha(L\{\tilde{S} \mid \tilde{S} \in T_k\}) = \alpha(L\{S \mid S \in T_{k-1}\}$ (for $k > 1$).

For $S \in T_k$ define T_S to be the set of

1. Components of S if S is disconnected (these are order-autonomous in S as a subset of either P or Q by Lemma 4.1),
2. Maximal S-order-autonomous subsets of S as in Lemma 4.16 if S is connected and has no nontrivial representation as a linear lexicographic sum,
3. Minimal S-chain-links as in Lemma 5.5 if S has a nontrivial representation as a linear lexicographic sum.

Let T_S^P be T_S with the order $\leqslant_P \cup =$, where $\leqslant_P$ is as induced by P via Definition 2.2 and let $=$ be equality of sets. Let T_S^Q be T_S with the order $\leqslant_Q \cup =$ where $\leqslant_Q$ is as induced by Q via Definition 2.2 and $=$ is equality of sets. If $T_S^P \neq T_S^Q$ (which cannot happen in case 1), then by Lemmas 4.16, 5.5 and 6.1 in case 2, the sets must be duals of each other. In this case reverse the order on S to obtain S', otherwise let $S' := S$. In case 3, there are finitely many ordered sets such that $T_S^P = S_1 \oplus S_2 \oplus \cdots S_n$ (with n bounded by the height of P) and a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that the order on T_S^Q is $S_{\sigma(1)} < S_{\sigma(2)} < \cdots < S_{\sigma(n)}$. There is a finite sequence of $\leqslant n$ dualizations of order-autonomous subsets of the form $S_{k_1} \oplus \cdots \oplus \cdots S_{k_l}$ that transforms the order on T_S^P to the order on T_S^Q (essentially a 'bubble sort' variation/bastardization). This sequence will assign to every S_k the order induced on it by P or its dual. In this case we let S' be S equipped with the order obtained in the above fashion. Let $T_k' := \{S' : S \in T_k\}$. By assumption $\alpha(L\{S \mid S \in T_k\}) = \alpha(L\{S' \mid S' \in T_k'\})$. (The dualizations to sort the pieces with a nontrivial linear lexicographic sum decomposition can be done simultaneously for all pieces in question.) Define $T_{S'}$ in the same fashion as T_S and let $T_{k+1} := \bigcup_{S' \in T_k'} T_{S'}$. Clearly, 1 is satisfied. Since the $S \in T_k$ were P- and Q-order-autonomous and since the $\tilde{S} \in T_{S'}$ are P- and Q-order-autonomous subsets of S', the $\tilde{S}$ are P- and Q-order-autonomous. Since P and Q have the same comparability graph every $\tilde{S} \in T_{S'}$ has the same comparability graph as a subset of P as when regarded as a subset of Q. Hence the $\tilde{S}$ satisfy 2. Since all that happened to the order on the $\tilde{S} \in T_{k+1}$ is at most a finite sequence of dualizations we have 3. Order T_{k+1} via $\tilde{S}_1 \sqsubseteq \tilde{S}_2$ iff ($\tilde{S}_1 \in T_{S_1'} \neq T_{S_2'} \ni \tilde{S}_2$ and $S_1 <_{T_k} S_2$) or ($\tilde{S}_1, \tilde{S}_2 \in T_{S'}$ and $\tilde{S}_1 \leqslant_{T_{S'}} \tilde{S}_2$). By construction of T_{k+1} this order is the same as the order induced on T_{k+1} by Q, hence we have 4. Moreover it is easy to see that $L\{\tilde{S} \mid \tilde{S} \in T_{k+1}\} = L\{S' \mid S' \in T'_k\}$, and hence $\alpha(L\{\tilde{S} \mid \tilde{S} \in T_{k+1}\}) = \alpha(L\{S \mid S \in T_k\})$, which is 5. Since P is of finite height, this construction must reach a stage in which each $S \in T_n$ is a singleton in finitely many steps. Then $Q = L\{S \mid S \in T_n\}$ and we are done. □

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